

A NOTE ON SUBORDINATION

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ABSTRACT. In this paper we present a result about analytic functions f defined on the open unit disc and with a finite number of exceptional values contained in the real interval $(0, 1)$. We find an upper bound for $|f'(0)|$. This bound is sharp in the case of one exceptional value. We also analyzed the case of two exceptional values.

1. PRELIMINARIES

The definition and results of this section can be seen in [1] (p. 450).

Definition 1. Let $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ be the open unit disk in the complex plane. Let f and g be analytic functions on \mathbb{D} . The function f is said to be subordinate to g (and we denote this relation by $f \prec g$) if there exists an analytic function $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ satisfying $\varphi(0) = 0$ and such that $f = g \circ \varphi$.

The following two lemmas are immediate consequences of the Definition 1.

Lemma 1. If $f \prec g$, then $|f'(0)| \leq |g'(0)|$.

Proof. We have $|f'(0)| = |g'(0)| |\varphi'(0)|$ and (by Schwarz's Lemma) $|\varphi'(0)| \leq 1$. \square

Lemma 2. Denote by \mathcal{P} the collection of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that $f(0) = 1$ and $\operatorname{Re}(f) > 0$. If $f \in \mathcal{P}$ and $F : \mathbb{D} \rightarrow \{z \in \mathbb{C}; \operatorname{Re}(z) > 0\}$ is the function given by

$$F(z) = \frac{1+z}{1-z}$$

then $f \prec F$.

Proof. It is sufficient to define $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ by $\varphi(z) = F^{-1}(f(z)) = \frac{f(z)-1}{f(z)+1}$. \square

2. MAIN RESULTS

Theorem 1. Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be an analytic function with $f(0) = 0$. If there exists real numbers $\{\alpha_1, \alpha_2, \dots, \alpha_k\} \subset (0, 1)$ such that $f(\mathbb{D}) \cap \{\alpha_1, \alpha_2, \dots, \alpha_k\} = \emptyset$. Then

$$(2.1) \quad |f'(0)| \leq \frac{2 \ln\left(\frac{1}{\alpha_1 \alpha_2 \dots \alpha_k}\right)}{\left[\frac{1-\alpha_1^2}{\alpha_1} + \dots + \frac{1-\alpha_k^2}{\alpha_k}\right]}$$

Proof. We define the analytic function $h : \mathbb{D} \rightarrow \mathbb{C} - \{0\}$ given by the following expression

$$h(z) = \left(\frac{\alpha_1 - f(z)}{1 - \alpha_1 f(z)}\right) \cdot \left(\frac{\alpha_2 - f(z)}{1 - \alpha_2 f(z)}\right) \cdot \dots \cdot \left(\frac{\alpha_k - f(z)}{1 - \alpha_k f(z)}\right)$$

The open disk \mathbb{D} is simply connected and then there exists an analytic function $g : \mathbb{D} \rightarrow \mathbb{C}$ such that $e^{g(z)} = h(z)$ for all $z \in \mathbb{D}$. Taking modulus on both sides of $e^g = h$ we obtain

$$e^{\operatorname{Re}(g(z))} = \left| \frac{\alpha_1 - f(z)}{1 - \alpha_1 f(z)} \right| \cdot \dots \cdot \left| \frac{\alpha_k - f(z)}{1 - \alpha_k f(z)} \right|, \quad \forall z \in \mathbb{D}$$

If we take the logarithms of both sides of the last expression, we obtain

$$\operatorname{Re}(g(z)) = \ln \left\{ \left| \frac{\alpha_1 - f(z)}{1 - \alpha_1 f(z)} \right| \cdot \dots \cdot \left| \frac{\alpha_k - f(z)}{1 - \alpha_k f(z)} \right| \right\}, \quad \forall z \in \mathbb{D}$$

We note that the expression for $\operatorname{Re}(g(z))$ is negative for all $z \in \mathbb{D}$. Define now

$$\rho : \mathbb{D} \rightarrow \mathbb{C}, \quad \rho(z) = \frac{g(z)}{\ln(\alpha_1) + \dots + \ln(\alpha_k)}$$

We claim that $\rho \in \mathcal{P}$. In fact, it's easy to see that the function ρ is analytic, $\rho(0) = 1$ and $\operatorname{Re}(\rho) > 0$. Hence, according to Lemma 2, we have $\rho \prec F$. The Lemma 1 gives us $|\rho'(0)| \leq |F'(0)| = 2$. Now, we take the derivative of both sides of $e^g = h$. We obtain

$$\begin{aligned} g'(z) &= e^{-g(z)} \cdot \left\{ \sum_{j=1}^k \left(\frac{\alpha_1 - f(z)}{1 - \alpha_1 f(z)} \right) \cdot \dots \cdot \left(\frac{\alpha_j - f(z)}{1 - \alpha_j f(z)} \right)' \cdot \dots \cdot \left(\frac{\alpha_k - f(z)}{1 - \alpha_k f(z)} \right) \right\} \\ &= e^{-g(z)} \cdot \left\{ \sum_{j=1}^k \left(\frac{\alpha_1 - f(z)}{1 - \alpha_1 f(z)} \right) \cdot \dots \cdot \left(\frac{(\alpha_j^2 - 1)f'(z)}{(1 - \alpha_j f(z))^2} \right) \cdot \dots \cdot \left(\frac{\alpha_k - f(z)}{1 - \alpha_k f(z)} \right) \right\} \end{aligned}$$

and then $g'(0) = f'(0) \cdot \left\{ \frac{\alpha_1^2 - 1}{\alpha_1} + \dots + \frac{\alpha_k^2 - 1}{\alpha_k} \right\}$ (because of $e^{g(0)} = \alpha_1 \cdot \dots \cdot \alpha_k$). Then,

$$|\rho'(0)| = \frac{|f'(0)|}{|\ln(\alpha_1) + \dots + \ln(\alpha_k)|} \cdot \left[\frac{1 - \alpha_1^2}{\alpha_1} + \dots + \frac{1 - \alpha_k^2}{\alpha_k} \right]$$

Then the inequality (2.1) is a consequence of $|\rho'(0)| \leq 2$. \square

Corollary 1. [2, p. 233] *Suppose that $\alpha \in (0, 1)$. If $f : \mathbb{D} \rightarrow \mathbb{D}$ is an analytic function with $f(0) = 0$ and $f(z) \neq \alpha$ for all $z \in \mathbb{D}$, then we have the following inequality $|f'(0)| \leq 2\alpha \ln(1/\alpha)/(1 - \alpha^2)$. This inequality is sharp in the sense that there exists a function f such that equality holds.*

Proof. The first part of the corollary is obvious. It follows from our proof of Theorem 1 that the equality occurs when $|\rho'(0)| = 2$. It is equivalent to say that $|\varphi'(0)| = 1$, where $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is the function $\varphi(z) = F^{-1}(\rho(z)) = \frac{g(z) - \ln(\alpha)}{g(z) + \ln(\alpha)}$. By Schwarz's Lemma, $|\varphi'(0)| = 1$ if and only if there exists a complex c such that $|c| = 1$ and $\varphi(z) = cz$. Then,

$$\begin{aligned} |\varphi'(0)| = 1 &\iff \varphi(z) = cz \iff g(z) = \left(\frac{1 + cz}{1 - cz} \right) \cdot \ln(\alpha) \\ &\Rightarrow \frac{\alpha - f(z)}{1 - \alpha f(z)} = e^{g(z)} = e^{(\frac{1+cz}{1-cz}) \ln(\alpha)} \\ &\Rightarrow f(z) = \frac{\alpha - e^{(\frac{1+cz}{1-cz}) \ln(\alpha)}}{1 - \alpha e^{(\frac{1+cz}{1-cz}) \ln(\alpha)}}, \quad \forall z \in \mathbb{D} \end{aligned}$$

The function f satisfies $f(0) = 0$, $|f| < 1$ and $|f'(0)| = 2\alpha \ln(1/\alpha)/(1 - \alpha^2)$. \square

We analyze now the case of two distinct exceptional values $\alpha_1, \alpha_2 \in (0, 1)$. With the notations of Corollary 1, the equality in (2.1) occurs if and only if

$$g(z) = \left[\frac{1+cz}{1-cz} \right] \ln(\alpha_1 \alpha_2)$$

Using the identity $h = e^g$ (where h is as in Theorem 1), we can write

$$\left(\frac{\alpha_1 - f(z)}{1 - \alpha_1 f(z)} \right) \cdot \left(\frac{\alpha_2 - f(z)}{1 - \alpha_2 f(z)} \right) = e^{\left[\frac{1+cz}{1-cz} \right] \ln(\alpha_1 \alpha_2)}$$

By expanding the above expression, we obtain

$$(1 - \alpha_1 \alpha_2 u) f^2 + (u - 1)(\alpha_1 + \alpha_2) f + \alpha_1 \alpha_2 - u = 0$$

where $u : \mathbb{D} \longrightarrow \mathbb{D} - \{0\}$ is the function $u(z) = e^{\left[\frac{1+cz}{1-cz} \right] \ln(\alpha_1 \alpha_2)}$.

Solving this equation in f , we find

$$f = \frac{(1-u)(\alpha_1 + \alpha_2) - \sqrt{(u-1)^2(\alpha_1 + \alpha_2)^2 - 4(1 - \alpha_1 \alpha_2 u)(\alpha_1 \alpha_2 - u)}}{2(1 - \alpha_1 \alpha_2 u)}$$

We define now the function $F : \mathbb{D} \longrightarrow \mathbb{C}$ by

$$\begin{aligned} F(z) &= (u(z) - 1)^2(\alpha_1 + \alpha_2)^2 - 4(1 - \alpha_1 \alpha_2 u(z))(\alpha_1 \alpha_2 - u(z)) \\ &= ((\alpha_1 - \alpha_2)^2)u(z)^2 + (4 + 4\alpha_1^2\alpha_2^2 - 2(\alpha_1 + \alpha_2)^2)u(z) + (\alpha_1 - \alpha_2)^2 \end{aligned}$$

Then f represents an analytic function if and only if $F(\mathbb{D})$ is in the domain of some branch of the logarithm. In general, this does not occur. For instance, if $\alpha_1 = 1/2$ and $\alpha_2 = 1/4$, $F(\mathbb{D})$ contains a neighborhood of origin.

But, if for a given choice of the constants α_1 and α_2 we have $F(\mathbb{D}) \cap [0, \infty) = \emptyset$, then there exists \sqrt{F} and f is analytic. To achieve this, it is sufficient that the solutions of $F(w) = t$ (with $t \geq 0$) be complex numbers with $|w| \geq 1$.

First, we observe that $F(w) = t$ iff

$$(2.2) \quad w = \frac{b - \sqrt{(-b)^2 - 4(\alpha_1 - \alpha_2)^2((\alpha_1 - \alpha_2)^2 - t)}}{2(\alpha_1 - \alpha_2)^2}$$

where $b := 4\alpha_1^2\alpha_2^2 + 2(\alpha_1 + \alpha_2)^2 - 4$. Suppose now that β is a real number in $(0, 1)$ such that $4\beta^4 + 8\beta^2 - 4 < 0$. Making $\alpha_1 \longrightarrow \beta^+$ and $\alpha_2 \longrightarrow \beta^-$, the expression (2.2) tends to infinity and then, $|w| > 1$ for all values α_1 and α_2 sufficiently close to β .

We can summarize our analysis in the following theorem

Theorem 2. *Let $\beta \in (0, 1)$ be a real number such that $4\beta^4 + 8\beta^2 - 4 < 0$. Consider distinct real numbers $\alpha_1, \alpha_2 \in (0, 1)$ sufficiently close to β . If $f : \mathbb{D} \longrightarrow \mathbb{D}$ is an analytic function with $f(0) = 0$ and $f(\mathbb{D}) \cap \{\alpha_1, \alpha_2\} = \emptyset$, then f satisfies the inequality (2.1) (with $k = 2$). This inequality is sharp in the sense that there exists a function f such that equality holds.*

REFERENCES

- [1] T. H. MACGREGOR, Geometric problems in complex analysis, *Amer. Math. Monthly* (1972), 447–468.
- [2] Z. NEHARI, *Conformal Mapping*, Dover (1975).

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